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GENERALIZED LAPLACE TRANSFORM AND FRACTIONAL EQUATIONS OF DISTRIBUTED ORDERS

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ABSTRACT

In this article, we introduce a generalized Laplace transform and fractional equations of distributed orders and evaluate the results of the complex inversion formula for the exponential Mellin transform, the exponential Laplace transform of δ_x-Derivatives.

KEYWORDS: Mellin Transform, Fractional Derivatives, Fractional Diffusion Equation

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1. INTRODUCTION

This section is devoted to a presentation of some basic facts from the theory of the Mellin integral transform that are used in the further discussions. For more information regarding the Mellin integral transform including its properties and particular cases we refer the interested reader to e.g. [2], [3], [5].

The Mellin integral transform of a sufficiently well-behaved function f is defined as

$$M\{f(t); s\} = f^*(s) = \int_0^{+\infty} f(t)t^{s-1}dt \quad (2.1)$$

and the inverse Mellin integral transform as

$$f(t) = M^{-1}\{f(t);s\}$$

$$= f^*(s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)t^{-s}ds, \; t > 0, \gamma = \Re(s) \quad (2.2)$$

Where the integral is understood in the sense of the Cauchy principal value.

It is worth mentioning that the Mellin integral transform can be obtained from the Fourier-integral transform by the variables substitution t=e^x and by rotation of the complex plane by a right angle:

$$M\{f(t); s\} = f^*(s) = \int_0^{+\infty} f(t)t^{s-1}dt = \int_{-\infty}^{+\infty} f(e^x)\, e^{ix(-is)}dx = \xi\{f(e^x)\, e^{ix(-is)}dx$$

$$= \xi\{fe^{x);-is}\},$$

Where ξ $\{f(x); \kappa\}$ denotes the Fourier transform of the function f at the point κ. Accordingly, the inverse Mellin transforms and the convolution for the Mellin transform can be obtained by the same substitutions from the inverse Fourier transform and the convolution of the Fourier transform. The integral in the right-hand side of (2.1) is well defined, e.g. for the functions $f \in Lc(,E)$, $0 < \_ < E < \infty$ continuous in the intervals (0, _] , $[E,+\infty)$ and satisfying the estimates $|f(t)| \le M$ $t-\gamma 1$ for $0 < t < \_$, $|f(t)| \le M\ t-\gamma 2$ for $t > E$, where M is a constant and $\gamma 1 < \gamma 2$. If these conditions hold true, the Mellin transform f?(s) exists and is an analytical function in the vertical strip $\gamma 1 < \_(s) < \gamma 2$. If a function f is piecewise differentiable, $f(t)\ t\gamma-1 \in Lc(0,+\infty)$, and its Mellin integral transform f?(s) is given by (2.1) then the formula (2.2) holds true at all points, where the function f is continuous.

The *Mellin Convolution*

$$(f^{M} * g)(x) = \int_0^{+\infty} f\left(\frac{x}{t}\right) g(t)\frac{dt}{t} \tag{2.3}$$

plays a very essential role in the further discussions. It is well known (see e.g. [35]) that if $f(t)\ t\gamma-1 \in L(0,\infty)$ and $g(t)\ t\gamma-1 \in L(0,\infty)$ then the Mellin convolution $h = (f^{M} * g)$ given by (2.3) is well defined, satisfies the important property

$$M\{(f \overset{m}{*} g)(x); s\} = m\{f(t);s\}.\ M\{g(t);s\}, \tag{2.4}$$

And $h(x)x^{\gamma-1} \in L(o,\infty)$. Morever, the Parseval equality

$$\int_0^{+\infty} f\left(\frac{x}{t}\right) g(t)\frac{dt}{t} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^{*}(s) g^{*}(s) x^{-s}\, ds \tag{2.5}$$

hold true.

Theorem 2.1 (The Complex Inversion Formula for the Exponential Mellin Transform)

Let F(P) be an analytic function of p (assuming that F(P) has not the has not the branch point) except a finite number of poles and each of poles lies to the left of the vertical line $\Re p = c$. If $F(p) \to 0$ as $p \to \infty$ through the left plane $\Re p \le c$, and

$$M\{f(x);P\} = F(P) = \int_0^{\infty} x^{p-1} f(x)dx$$

$$M^{-1}\{F(P)\} = f(x) = \frac{1}{2\pi i}\int_{c=i\infty}^{c+i\infty} F(P)x^{-P}dp$$

Proof: By definition of the exponential Laplace transform (1-1) and letting P = r, we

Have

$$F(r) = \int_0^{\infty} x^{p-1} f(x)dx$$

now, by setting t = p -1 in the above relation, we obtain

$$F(r) = \int_0^{\infty} x^{t} f(t)dt$$

$$= M\{f(t);r\}$$

At this point, by the complex inversion formula for the Mellin transform and setting back

t-x,r=P, we get finally

f(x) = $\frac{1}{2\pi i}\int_{c=i\infty}^{c+i\infty} F(P)x^{-P}$dp

Theorem 2.2 (The Exponential Laplace Transform of δ_x-Derivatives)

Let f; f',….. f^{n-1} are continuous functions with piecewise continuous derivative $f^{(n)}$ on the interval x ≥0 and if all functions are of exponential order x^{p-1} as x→∞ (i.e. jf(x)≤ Mx^{p-1} for some constants c; M, then for n = 1,2,……

M{$\delta_x^n f(x): P$} = $(-P)^n$F(P)M{f(x):P}-$(-P)^{n-1}$F(P)f(0^+) - $(-P)^{n-2}$F(P)(δ_xf)(0^+) - ……….- (δ_x^{n-1}f)(0^+)

where the δ_x-derivative operator is defined as follows

δ_x = x$\frac{d}{dx}$

And by notation

δ_x^2 = (δ_x)(δ_x)

$= x^2\frac{d^2}{dx^2}$

$= x^2\frac{d^2}{dx^2} + \frac{x''}{x'^3}\frac{d}{dx}$

The δ_x -derivative for any positive integer power can be found.

Proof: Using the definitions of the exponential Mellin transform (1.1) and the δ_x –derivative, by integration by parts, we obtain

M{$\delta_x f(x); P$} = $\int_0^\infty x^{p-1}f'(x)$dx = x^{p-1}f(x)$|_0^\infty$+(p -1)$\int_0^\infty x^{p-2}f''(x)$dx

Since f is of exponential order x^{p-1} as x→∞. follows that

$\lim_{x\to\infty} x^{p-1}f(x) = 0$

Consequently

M{δ_xf(x):P} = $x^{p-1}M\{f(x);P\} - f(0^+)$

Similarly, by repeated application of the above relation once again, we get

M{δ_x^2f(x):P} = $x^{p-1}M\{\delta_x$f(x):P)- $(\delta_x f)(0^+)$

= $x^{p-2}M\{f(x):P\} - x^{p-1}f(0^+) - (\delta_x f)(0^+)$

And by repeating the above scheme for δ_x^nf(x).

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